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Fayet-Iliopoulos *D* term and its renormalization in the minimal supersymmetric standard model

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We consider the renormalization of the Fayet-Iliopoulos D term in a softly broken supersymmetric gauge theory with a nonsimple gauge group containing an Abelian factor, and present the associated β function through three loops. We also include in an appendix the result for several Abelian factors. We specialize to the case of the minimal supersymmetric standard model, and investigate the behavior of the Fayet-Iliopoulos coupling ξ for various boundary conditions at the unification scale. We focus particularly on the case of nonstandard soft supersymmetry breaking couplings, for which ξ evolves significantly between the unification scale and the weak scale.

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I. INTRODUCTION

In Abelian gauge theories with $N=1$ supersymmetry there exists a possible invariant that is not allowed in the non-Abelian case: the Fayet-Iliopoulos D term:

$$L=\xi\int V(x,\theta,\overline{\theta})\, d^4\theta=\xi D(x). \tag{1.1}$$

In previous papers [1,2] we have discussed the renormalization of ξ in the presence of the standard soft supersymmetry-breaking terms

$$L_{\mathrm{SB}}=-(m^2)^j_i\phi^i\phi_j -\left(\frac{1}{6}h^{ijk}\phi_i\phi_j\phi_k+\frac{1}{2}b^{ij}\phi_i\phi_j+\frac{1}{2}M\lambda\lambda+\text{H.c.}\right). \tag{1.2}$$

The result for β_ξ is as follows:

$$\beta_\xi=\frac{\beta_g}{g}\xi+\hat{\beta}_\xi \tag{1.3}$$

where $\hat{\beta}_\xi$ is determined by V-tadpole (or in components D-tadpole) graphs, and is independent of ξ. Although in Refs. [1,2] we restricted ourselves to the Abelian case, it is evident that a D term can occur with a direct product gauge group ($G_1\otimes G_2\cdots$) if there is an Abelian factor: as is the case for the minimal supersymmetric standard model (MSSM). In the MSSM context one may treat ξ as a free parameter at the weak scale [3], in which case there is no need to know $\hat{\beta}_\xi$. However, if we know ξ at gauge unification, for example, then we need $\hat{\beta}_\xi$ to predict ξ at low energies. Our purpose in this paper is first of all to give the result for $\hat{\beta}_\xi$ through three loops for a general direct product gauge group. For simplicity of exposition, we restrict ourselves in the main body of the paper to the case of one Abelian factor, postponing the more general result (which is complicated by the possibility of ‘‘kinetic mixing’’ [4] between different Abelian factors) to an appendix. We shall then specialize to the case of the MSSM, and perform some running analyses to determine the size of $\xi(M_Z)$ for various choices of boundary conditions at the unification scale M_X.

II. GENERAL CASE

First of all, for completeness and to establish the notation, let us recapitulate the standard results for supersymmetric theory. We take an $N=1$ supersymmetric gauge theory with gauge group $\Pi_\alpha G_\alpha$ and with superpotential

$$W(\Phi)=\frac{1}{6}Y^{ijk}\Phi_i\Phi_j\Phi_k+\frac{1}{2}\mu^{ij}\Phi_i\Phi_j. \tag{2.1}$$

We will be assuming here that the gauge group has one Abelian factor, which we shall take to be G_1. We shall denote the hypercharge matrix for G_1 by $\mathcal{Y}$. At one loop we have

$$16\pi^2\beta^{(1)}_{g_\alpha}=g^3_\alpha Q_\alpha=g^3_\alpha[T(R_\alpha)-3C(G_\alpha)], \tag{2.2a}$$

$$16\pi^2\gamma^{(1)i}{}_j=P^i{}_j=\frac{1}{2}Y^{ikl}Y_{jkl}-2\sum_\alpha g^2_\alpha[C(R_\alpha)]^i{}_j, \tag{2.2b}$$

where R_α is the group representation for G_α acting on the chiral fields, $C(R_\alpha)$ the corresponding quadratic Casimir, and $T(R_\alpha)=(r_\alpha)^{-1}\mathrm{Tr}[C(R_\alpha)]$, r_α being the dimension of G_α. For the adjoint representation, $C(R_\alpha)=C(G_\alpha)I_\alpha$, where I_α is the $r_\alpha\times r_\alpha$ unit matrix. Note that $T(R_1)=\mathrm{Tr}[\mathcal{Y}^2]$, $[C(R_1)]^i{}_j=(\mathcal{Y}^2)^i{}_j$. At two loops we have

$$(16\pi^2)^2\beta^{(2)}_{g_\alpha}=2g^5_\alpha C(G_\alpha)Q_\alpha-2g^3_\alpha r^{-1}_\alpha\mathrm{Tr}[PC(R_\alpha)] \tag{2.3a}$$

$$(16\pi^2)^2\gamma^{(2)i}{}_j=\left[-Y_{jmn}Y^{mpi}-2\sum_\alpha g^2_\alpha C(R_\alpha)^p{}_j\delta^i{}_n\right]P^n{}_p +2\sum_\alpha g^4_\alpha C(R_\alpha)^i{}_jQ_\alpha. \tag{2.3b}$$

For completeness and later reference, we also quote here the general result for $\beta^{\mathrm{DRED}(3)}_{g_\alpha}$, which is a straightforward generalization of the result of Ref. [5]:

$$\beta_{g_\alpha}^{\mathrm{DRED}(3)} = 3r_\alpha^{-1}g_\alpha^3 Y^{ikm}Y_{jkn}P^n{}_m C(R_\alpha)^j{}_i + 6r_\alpha^{-1}g_\alpha^3\sum_\beta g_\beta^2 \mathrm{Tr}[PC(R_\alpha)C(R_\beta)] + r_\alpha^{-1}g_\alpha^3\mathrm{Tr}[P^2C(R_\alpha)] - 6r_\alpha^{-1}g_\alpha^3\sum_\beta Q_\beta g_\beta^4\mathrm{Tr}[C(R_\alpha)C(R_\beta)] - 4r_\alpha^{-1}g_\alpha^5 C(G_\alpha)\mathrm{Tr}[PC(R_\alpha)] + g_\alpha^7 Q_\alpha C(G_\alpha)[4C(G_\alpha) - Q_\alpha]. \quad (2.4)$$

We recall that gauge anomaly cancellation requires

$$\mathrm{Tr}[\mathcal{Y}C(R_\alpha)] = 0 \quad (2.5)$$

and naturalness (or cancellation of U_1-gravitational anomalies) requires

$$\mathrm{Tr}[\mathcal{Y}] = 0. \quad (2.6)$$

The diagrams contributing to $\hat{\beta}_\xi$ through three loops for a general non-simple gauge group are essentially the same as those depicted for the pure Abelian case in Ref. [2], but reinterpreting internal gauge and gaugino propagators as ranging over all gauge groups in the direct product. Potential new 3-loop graphs (involving a 3-point gauge vertex or a gauge-gaugino vertex) give contributions which vanish due to anomaly cancellation (such as $C(G_\alpha)\mathrm{Tr}[\mathcal{Y}C(R_\alpha)]$). It is then relatively easy to generalize the Abelian result to the general case. We find

$$16\pi^2\hat{\beta}_\xi^{(1)} = 2g_1\mathrm{Tr}[\mathcal{Y}m^2] \quad (2.7)$$

$$16\pi^2\hat{\beta}_\xi^{(2)} = -4g_1\mathrm{Tr}[\mathcal{Y}m^2\gamma^{(1)}], \quad (2.8)$$

$$(16\pi^2)^3\frac{\hat{\beta}_\xi^{(3)\mathrm{DRED}'}}{g_1} = -6(16\pi^2)^2\mathrm{Tr}[\mathcal{Y}m^2\gamma^{(2)}] - 4\mathrm{Tr}[\mathcal{Y}WP] - \frac{5}{2}\mathrm{Tr}[\mathcal{Y}HH^\dagger] + 2\mathrm{Tr}[\mathcal{Y}P^2m^2] - 24\zeta(3)\sum_\alpha g_\alpha^2\mathrm{Tr}[\mathcal{Y}WC(R_\alpha)] + 12\zeta(3)\sum_\alpha g_\alpha^2\mathrm{Tr}[\mathcal{Y}M_\alpha^* HC(R_\alpha) + \mathrm{c.c.}] - 96\zeta(3)\sum_{\alpha,\beta} g_\alpha^2 g_\beta^2 M_\alpha M_\alpha^*\mathrm{Tr}[\mathcal{Y}C(R_\alpha)C(R_\beta)] - 24\zeta(3) \times\left\{\sum_{\alpha,\beta} g_\alpha^2 g_\beta^2 M_\alpha M_\beta^*\mathrm{Tr}[\mathcal{Y}C(R_\alpha)C(R_\beta)] + \mathrm{c.c.}\right\} \quad (2.9)$$

where [6]

$$W^i{}_j = \left(\frac{1}{2}Y^2m^2 + \frac{1}{2}m^2Y^2 + h^2\right)^i{}_j + 2Y^{ipq}Y_{jpr}(m^2)^r{}_q - 8\sum_\beta g_\beta^2 M_\beta M_\beta^* C(R_\beta)^i{}_j, \quad (2.10)$$

$$H^i{}_j = h^{ikl}Y_{jkl} + 4\sum_\beta g_\beta^2 M_\beta[C(R_\beta)]^i{}_j \quad (2.11)$$

with $(Y^2)^i{}_j = Y^{ikl}Y_{jkl}$, $(h^2)^i{}_j = h^{ikl}h_{jkl}$. These results are computed using the DRED′ scheme, which is a variant of the dimensional reduction scheme (DRED) defined so as to ensure that β functions for physical couplings have no dependence on the ϵ-scalar mass [7]. Most of the terms in Eq. (2.9) correspond in a simple way to the analogous terms in Eq. (5.2) of Ref. [2], the only subtle point being the MM^*g^4 terms, where one sees easily that only in the case of Fig. 15(e) of [2] can the two gaugino masses belong to different gauge groups (G_α). Thus the last term in Eq. (2.9) and the MM^*g^4 terms from the terms involving H come entirely from this particular figure.

It was proved in Ref. [1] in the pure Abelian case that if the m^2 dependence in $\hat{\beta}_\xi$ takes the form $\mathrm{Tr}[m^2A]$, then we have

$$\mathrm{Tr}[\mathcal{Y}A] = 2\frac{\beta_{g_1}}{g_1^2}. \quad (2.12)$$

It is easy to see that the proof extends to the direct product case, and indeed we can check Eq. (2.12) explicitly using Eqs. (2.7)–(2.9) and (2.2a), (2.3a) and (2.4).

III. MSSM

We now specialize to the case of the MSSM. The relevant part of the MSSM superpotential is

$$W = H_2 t^c Y_t Q + H_1 b^c Y_b Q + H_1 \tau^c Y_\tau L \quad (3.1)$$

where Y_t, Y_b, Y_τ are 3×3 Yukawa flavor matrices.

The gauge β functions are given at one loop by

$$16\pi^2\beta_{g_\alpha}=b_\alpha g_\alpha^3, \tag{3.2}$$

where

$$b_1=\tfrac{33}{5},\quad b_2=1,\quad b_3=-3, \tag{3.3}$$

and our U_1 coupling normalization corresponds to the usual one such that $g_1^2=\frac{5}{3}(g')^2$. For the anomalous dimensions of the chiral superfields we have, at one loop,

$$\begin{aligned}
16\pi^2\gamma_{t^c}^{(1)}&=P_{t^c}=2Y_tY_t^\dagger-2C_{t^c},\\
16\pi^2\gamma_{b^c}^{(1)}&=P_{b^c}=2Y_bY_b^\dagger-2C_{b^c},\\
16\pi^2\gamma_{Q}^{(1)}&=P_{Q}=Y_b^\dagger Y_b+Y_t^\dagger Y_t-2C_Q,\\
16\pi^2\gamma_{\tau^c}^{(1)}&=P_{\tau^c}=2Y_\tau Y_\tau^\dagger-2C_{\tau^c},\\
16\pi^2\gamma_{L}^{(1)}&=P_{L}=Y_\tau^\dagger Y_\tau-2C_L,\\
16\pi^2\gamma_{H_1}^{(1)}&=P_{H_1}=\mathrm{Tr}[Y_\tau^\dagger Y_\tau+3Y_b^\dagger Y_b]-2C_L,\\
16\pi^2\gamma_{H_2}^{(1)}&=P_{H_2}=3\mathrm{Tr}[Y_t^\dagger Y_t]-2C_L,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
C_{t^c}&=\tfrac{4}{3}g_3^2+\tfrac{4}{15}g_1^2,\\
C_{b^c}&=\tfrac{4}{3}g_3^2+\tfrac{1}{15}g_1^2,\\
C_Q&=\tfrac{4}{3}g_3^2+\tfrac{3}{4}g_2^2+\tfrac{1}{60}g_1^2,\\
C_{\tau^c}&=\tfrac{3}{5}g_1^2,\\
C_L&=\tfrac{3}{4}g_2^2+\tfrac{3}{20}g_1^2.
\end{aligned} \tag{3.5}$$

At two loops [8] the anomalous dimensions are given by

$$\begin{aligned}
(16\pi^2)^2\gamma_{t^c}^{(2)}&=-2Y_t(P_Q+P_{H_2})Y_t^\dagger-2P_{t^c}C_{t^c}\\
&\quad+2(\tfrac{4}{15}b_1g_1^4+\tfrac{4}{3}b_3g_3^4),
\end{aligned} \tag{3.6a}$$

$$\begin{aligned}
(16\pi^2)^2\gamma_{b^c}^{(2)}&=-2Y_b(P_Q+P_{H_1})Y_b^\dagger\\
&\quad-2P_{b^c}C_{b^c}+2(\tfrac{1}{15}b_1g_1^4+\tfrac{4}{3}b_3g_3^4),
\end{aligned} \tag{3.6b}$$

$$\begin{aligned}
(16\pi^2)^2\gamma_{Q}^{(2)}&=-Y_t^\dagger(P_{t^c}+P_{H_2})Y_t-Y_b^\dagger(P_{b^c}+P_{H_1})Y_b\\
&\quad-2P_QC_Q+2(\tfrac{1}{60}b_1g_1^4+\tfrac{3}{4}b_2g_2^4\\
&\quad+\tfrac{4}{3}b_3g_3^4),
\end{aligned} \tag{3.6c}$$

$$\begin{aligned}
(16\pi^2)^2\gamma_{\tau^c}^{(2)}&=-2Y_\tau(P_L+P_{H_1})Y_\tau^\dagger\\
&\quad-2P_{\tau^c}C_{\tau^c}+\tfrac{6}{5}b_1g_1^4,
\end{aligned} \tag{3.6d}$$

$$\begin{aligned}
(16\pi^2)^2\gamma_{L}^{(2)}&=-Y_\tau^\dagger[P_{\tau^c}+P_{H_1}]Y_\tau-2P_LC_L+\tfrac{3}{10}b_1g_1^4\\
&\quad+\tfrac{3}{2}b_2g_2^4,
\end{aligned} \tag{3.6e}$$

$$\begin{aligned}
(16\pi^2)^2\gamma_{H_1}^{(2)}&=-3\mathrm{Tr}[Y_bP_QY_b^\dagger+Y_b^\dagger P_{b^c}Y_b]\\
&\quad-\mathrm{Tr}[Y_\tau P_QY_\tau^\dagger+Y_\tau^\dagger P_{\tau^c}Y_\tau]-2C_LP_{H_1}\\
&\quad+\tfrac{3}{10}b_1g_1^4+\tfrac{3}{2}b_2g_2^4,
\end{aligned} \tag{3.6f}$$

$$\begin{aligned}
(16\pi^2)^2\gamma_{H_2}^{(2)}&=-3\mathrm{Tr}[Y_tP_QY_t^\dagger+Y_t^\dagger P_{t^c}Y_t]-2C_LP_{H_2}\\
&\quad+\tfrac{3}{10}b_1g_1^4+\tfrac{3}{2}b_2g_2^4.
\end{aligned} \tag{3.6g}$$

We now turn to the soft couplings. The quantities W and H defined in Eqs. (2.10), (2.11) are given by

$$\begin{aligned}
W_{t^c}&=(2m_{t^c}^2+4m_{H_2}^2)Y_tY_t^\dagger+4Y_tm_Q^2Y_t^\dagger\\
&\quad+2Y_tY_t^\dagger m_{t^c}^2+4h_th_t^\dagger-8C_{t^c}^{MM},\\
W_{b^c}&=(2m_{b^c}^2+4m_{H_1}^2)Y_bY_b^\dagger+4Y_bm_Q^2Y_b^\dagger+2Y_bY_b^\dagger m_{b^c}^2\\
&\quad+4h_bh_b^\dagger-8C_{b^c}^{MM},\\
W_Q&=(m_Q^2+2m_{H_2}^2)Y_t^\dagger Y_t+(m_Q^2+2m_{H_1}^2)Y_b^\dagger Y_b+[Y_t^\dagger Y_t\\
&\quad+Y_b^\dagger Y_b]m_Q^2+2Y_t^\dagger m_{t^c}^2Y_t+2Y_b^\dagger m_{b^c}^2Y_b+2h_t^\dagger h_t\\
&\quad+2h_b^\dagger h_b-8C_Q^{MM},\\
W_{\tau^c}&=(2m_{\tau^c}^2+4m_{H_1}^2)Y_\tau Y_\tau^\dagger+4Y_\tau m_L^2Y_\tau^\dagger+2Y_\tau Y_\tau^\dagger m_{\tau^c}^2\\
&\quad+4h_\tau h_\tau^\dagger-8C_{\tau^c}^{MM},\\
W_L&=(m_L^2+2m_{H_1}^2)Y_\tau^\dagger Y_\tau+2Y_\tau^\dagger m_{\tau^c}^2Y_\tau+Y_\tau^\dagger Y_\tau m_L^2+2h_\tau^\dagger h_\tau\\
&\quad-8C_L^{MM},\\
W_{H_1}&=\mathrm{Tr}[6(m_{H_1}^2+m_Q^2)Y_b^\dagger Y_b+6m_{b^c}^2Y_bY_b^\dagger\\
&\quad+2(m_{H_1}^2+m_L^2)Y_\tau^\dagger Y_\tau+2Y_\tau^\dagger m_{\tau^c}^2Y_\tau\\
&\quad+6h_b^\dagger h_b+2h_\tau^\dagger h_\tau]-8C_L^{MM},\\
W_{H_2}&=6\mathrm{Tr}[(m_{H_2}^2+m_Q^2)Y_t^\dagger Y_t+m_{t^c}^2Y_tY_t^\dagger+h_t^\dagger h_t]-8C_L^{MM},
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
C_{t^c}^{MM}&=\tfrac{4}{3}|M_3|^2g_3^2+\tfrac{4}{15}|M_1|^2g_1^2,\\
C_{b^c}^{MM}&=\tfrac{4}{3}|M_3|^2g_3^2+\tfrac{1}{15}|M_1|^2g_1^2,\\
C_{Q}^{MM}&=\tfrac{4}{3}|M_3|^2g_3^2+\tfrac{3}{4}|M_2|^2g_2^2+\tfrac{1}{60}|M_1|^2g_1^2,\\
C_{\tau^c}^{MM}&=\tfrac{3}{5}|M_1|^2g_1^2,\\
C_{L}^{MM}&=\tfrac{3}{4}|M_2|^2g_2^2+\tfrac{3}{20}|M_1|^2g_1^2
\end{aligned} \tag{3.8}$$

and

$$
\begin{aligned}
H_{t^c}&=4h_tY_t^{\dagger}+4C_{t^c}^M,\\
H_{b^c}&=4h_bY_b^{\dagger}+4C_{b^c}^M,\\
H_Q&=2(Y_t^{\dagger}h_t+Y_b^{\dagger}h_b)+4C_Q^M,\\
H_{\tau^c}&=4h_\tau Y_\tau^{\dagger}+4C_{\tau^c}^M,\\
H_L&=2Y_\tau^{\dagger}h_\tau+4C_L^M,\\
H_{H_1}&=\mathrm{Tr}[6Y_b^{\dagger}h_b+Y_\tau^{\dagger}h_\tau]+4C_L^M,\\
H_{H_2}&=6\,\mathrm{Tr}[Y_t^{\dagger}h_t]+4C_L^M,
\end{aligned} \tag{3.9}
$$

where

$$
\begin{aligned}
C_{t^c}^M&=\tfrac{4}{3}M_3g_3^2+\tfrac{4}{15}M_1g_1^2,\\
C_{b^c}^M&=\tfrac{4}{3}M_3g_3^2+\tfrac{1}{15}M_1g_1^2,\\
C_Q^M&=\tfrac{4}{3}M_3g_3^2+\tfrac{3}{4}M_2g_2^2+\tfrac{1}{60}M_1g_1^2,\\
C_{\tau^c}^M&=\tfrac{3}{5}M_1g_1^2,\\
C_L^M&=\tfrac{3}{4}M_2g_2^2+\tfrac{3}{20}M_1g_1^2.
\end{aligned} \tag{3.10}
$$

With all these subsidiary definitions we can now give the results for $\hat{\beta}_\xi$ up to three loops. We have

$$
\begin{aligned}
16\pi^2\hat{\beta}_\xi^{(1)}=2\sqrt{\tfrac{3}{5}}g_1\mathrm{Tr}[&m_Q^2-m_L^2-2m_{t^c}^2\\
&+m_{b^c}^2+m_{\tau^c}^2-m_{H_1}^2+m_{H_2}^2]
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
(16\pi^2)^2\hat{\beta}_\xi^{(2)}=-4\sqrt{\tfrac{3}{5}}g_1\mathrm{Tr}(&m_Q^2P_Q-m_L^2P_L\\
&-2m_{t^c}^2P_{t^c}+m_{b^c}^2P_{b^c}+m_{\tau^c}^2P_{\tau^c}\\
&-m_{H_1}^2P_{H_1}+m_{H_2}^2P_{H_2})
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
(16\pi^2)^3\hat{\beta}_\xi^{(3)}=\sqrt{\tfrac{3}{5}}g_1[&-6(16\pi^2)^2\beta_{\xi_1}^{(3)}\\
&-4\beta_{\xi_2}^{(3)}-\tfrac{5}{2}\beta_{\xi_3}^{(3)}\\
&+2\beta_{\xi_4}^{(3)}+\zeta(3)(-24\beta_{\xi_5}^{(3)}\\
&+12\beta_{\xi_6}^{(3)}-96\beta_{\xi_7}^{(3)}-48\beta_{\xi_8}^{(3)})],
\end{aligned} \tag{3.13}
$$

where

$$
\begin{aligned}
\beta_{\xi_1}^{(3)}&=\mathrm{Tr}(m_Q^2\gamma_Q^{(2)}-m_L^2\gamma_L^{(2)}-2m_{t^c}^2\gamma_{t^c}^{(2)}+m_{b^c}^2\gamma_{b^c}^{(2)}\\
&\quad+m_{\tau^c}^2\gamma_{\tau^c}^{(2)}-m_{H_1}^2\gamma_{H_1}^{(2)}+m_{H_2}^2\gamma_{H_2}^{(2)}),\\
\beta_{\xi_2}^{(3)}&=\mathrm{Tr}(W_QP_Q-W_LP_L-2W_{t^c}P_{t^c}+W_{b^c}P_{b^c}\\
&\quad+W_{\tau^c}P_{\tau^c}-W_{H_1}P_{H_1}+W_{H_2}P_{H_2}),\\
\beta_{\xi_3}^{(3)}&=\mathrm{Tr}(H_Q^{\dagger}H_Q-H_L^{\dagger}H_L-2H_{t^c}^{\dagger}H_{t^c}+H_{b^c}^{\dagger}H_{b^c}\\
&\quad+H_{\tau^c}^{\dagger}H_{\tau^c}-H_{H_1}^{\dagger}H_{H_1}+H_{H_2}^{\dagger}H_{H_2}),\\
\beta_{\xi_4}^{(3)}&=\mathrm{Tr}(m_Q^2P_Q^2-m_L^2P_L^2-2m_{t^c}^2P_{t^c}^2+m_{b^c}^2P_{b^c}^2\\
&\quad+m_{\tau^c}^2P_{\tau^c}^2-m_{H_1}^2P_{H_1}^2+m_{H_2}^2P_{H_2}^2),\\
\beta_{\xi_5}^{(3)}&=\mathrm{Tr}(W_QC_Q-W_LC_L-2W_{t^c}C_{t^c}+W_{b^c}C_{b^c}\\
&\quad+W_{\tau^c}C_{\tau^c}-W_{H_1}C_L+W_{H_2}C_L),\\
\beta_{\xi_6}^{(3)}&=\mathrm{Tr}(H_QC_Q^{M*}-H_LC_L^{M*}-2H_{t^c}C_{t^c}^{M*}\\
&\quad+H_{b^c}C_{b^c}^{M*}+H_{\tau^c}C_{\tau^c}^{M*}-H_{H_1}C_L^{M*}\\
&\quad+H_{H_2}C_L^{M*})+\text{c.c.},\\
\beta_{\xi_7}^{(3)}&=3(C_Q^{MM}C_Q-C_L^{MM}C_L-2C_{t^c}^{MM}C_{t^c}\\
&\quad+C_{b^c}^{MM}C_{b^c}+C_{\tau^c}^{MM}C_{\tau^c})\\
\beta_{\xi_8}^{(3)}&=3(|C_Q^M|^2-|C_L^M|^2-2|C_{t^c}^M|^2+|C_{b^c}^M|^2\\
&\quad+|C_{\tau^c}^M|^2).
\end{aligned} \tag{3.14}
$$

We shall now present our MSSM results specialized to the commonly considered case where only the 3rd generation Yukawa couplings are significant. We also take the gaugino masses to be real. Writing λ_t, λ_b and λ_τ for the 3rd generation couplings, Eq. (3.4) becomes

$$
\begin{aligned}
P_{t^c}&=2\lambda_t^2-2C_{t^c}\\
P_{b^c}&=2\lambda_b^2-2C_{b^c}\\
P_Q&=\lambda_b^2+\lambda_t^2-2C_Q\\
P_{\tau^c}&=2\lambda_\tau^2-2C_{\tau^c}\\
P_L&=\lambda_\tau^2-2C_L\\
P_{u^c}&=-2C_{t^c}\\
P_{d^c}&=-2C_{b^c}\\
P_R&=-2C_Q\\
P_{e^c}&=-2C_{\tau^c}\\
P_N&=-2C_L\\
P_{H_1}&=\lambda_\tau^2+3\lambda_b^2-2C_L\\
P_{H_2}&=3\lambda_t^2-2C_L,
\end{aligned} \tag{3.15}
$$

where $\{t,b,Q,\tau,L\}$ now refers to the 3rd generation, and $\{u,d,R,e,N\}$ refers to either of the 1st or 2nd generation. Equation (3.6a) now takes the form

$$(16\pi^2)^2\gamma_{t^c}^{(2)} = -2\lambda_t^2(P_Q+P_{H_2}) - 2P_{t^c}C_{t^c} + 2(\tfrac{4}{15}b_1g_1^4 + \tfrac{4}{3}b_3g_3^4), \tag{3.16a}$$

$$(16\pi^2)^2\gamma_{b^c}^{(2)} = -2\lambda_b^2(P_Q+P_{H_1}) - 2P_{b^c}C_{b^c} + 2(\tfrac{1}{15}b_1g_1^4 + \tfrac{4}{3}b_3g_3^4), \tag{3.16b}$$

$$(16\pi^2)^2\gamma_Q^{(2)} = -\lambda_t^2(P_{t^c}+P_{H_2}) - \lambda_b^2(P_{b^c}+P_{H_1}) - 2P_QC_Q + 2(\tfrac{1}{60}b_1g_1^4 + \tfrac{3}{4}b_2g_2^4 + \tfrac{4}{3}b_3g_3^4), \tag{3.16c}$$

$$(16\pi^2)^2\gamma_{\tau^c}^{(2)} = -2\lambda_\tau^2(P_L+P_{H_1}) - 2P_{\tau^c}C_{\tau^c} + \tfrac{6}{5}b_1g_1^4, \tag{3.16d}$$

$$(16\pi^2)^2\gamma_L^{(2)} = -\lambda_\tau^2[P_{\tau^c}+P_{H_1}] - 2P_LC_L + \tfrac{3}{10}b_1g_1^4 + \tfrac{3}{2}b_2g_2^4, \tag{3.16e}$$

$$(16\pi^2)^2\gamma_{u^c}^{(2)} = -2P_{u^c}C_{t^c} + 2(\tfrac{4}{15}b_1g_1^4 + \tfrac{4}{3}b_3g_3^4), \tag{3.16f}$$

$$(16\pi^2)^2\gamma_{d^c}^{(2)} = -2P_{d^c}C_{b^c} + 2(\tfrac{1}{15}b_1g_1^4 + \tfrac{4}{3}b_3g_3^4), \tag{3.16g}$$

$$(16\pi^2)^2\gamma_R^{(2)} = -2P_RC_Q + 2(\tfrac{1}{60}b_1g_1^4 + \tfrac{3}{4}b_2g_2^4 + \tfrac{4}{3}b_3g_3^4), \tag{3.16h}$$

$$(16\pi^2)^2\gamma_{e^c}^{(2)} = -2P_{e^c}C_{\tau^c} + \tfrac{6}{5}b_1g_1^4, \tag{3.16i}$$

$$(16\pi^2)^2\gamma_N^{(2)} = -2P_NC_L + \tfrac{3}{10}b_1g_1^4 + \tfrac{3}{2}b_2g_2^4, \tag{3.16j}$$

$$(16\pi^2)^2\gamma_{H_1}^{(2)} = -3\lambda_b^2[P_Q+P_{b^c}] - \lambda_\tau^2[P_L+P_{\tau^c}] - 2C_LP_{H_1} + \tfrac{3}{10}b_1g_1^4 + \tfrac{3}{2}b_2g_2^4, \tag{3.16k}$$

$$(16\pi^2)^2\gamma_{H_2}^{(2)} = -3\lambda_t^2[P_Q+P_{t^c}] - 2C_LP_{H_2} + \tfrac{3}{10}b_1g_1^4 + \tfrac{3}{2}b_2g_2^4. \tag{3.16l}$$

Correspondingly, we retain only the three 3rd generation trilinear soft couplings $h_t = A_t\lambda_t$, $h_t = A_b\lambda_b$ and $h_\tau = A_\tau\lambda_\tau$. Equation (3.7) now becomes

$$\begin{aligned}
W_{t^c} &= 4\lambda_t^2(m_{t^c}^2 + m_Q^2 + m_{H_2}^2 + A_t^2) - 8C_{t^c}^{MM},\\
W_{b^c} &= 4\lambda_b^2(m_{b^c}^2 + m_Q^2 + m_{H_1}^2 + A_b^2) - 8C_{b^c}^{MM},\\
W_Q &= 2\lambda_t^2(m_{t^c}^2 + m_Q^2 + m_{H_2}^2 + A_t^2) + 2\lambda_b^2(m_{b^c}^2 + m_Q^2 + m_{H_1}^2 + A_b^2) - 8C_Q^{MM},\\
W_{\tau^c} &= 4\lambda_\tau^2(m_{\tau^c}^2 + m_{H_1}^2 + m_L^2 + A_\tau^2) - 8C_{\tau^c}^{MM},\\
W_L &= 2\lambda_\tau^2(m_{H_1}^2 + m_{\tau^c}^2 + m_L^2 + A_\tau^2) - 8C_L^{MM},\\
W_{u^c} &= -8C_{t^c}^{MM},\\
W_{d^c} &= -8C_{b^c}^{MM},\\
W_R &= -8C_Q^{MM},\\
W_{e^c} &= -8C_{\tau^c}^{MM},\\
W_N &= -8C_L^{MM},\\
W_{H_1} &= 6\lambda_b^2(m_{H_1}^2 + m_{b^c}^2 + m_Q^2 + A_b^2) + 2\lambda_\tau^2(m_{H_1}^2 + m_L^2 + m_{\tau^c}^2 + A_\tau^2) - 8C_L^{MM},\\
W_{H_2} &= 6\lambda_t^2(m_{H_2}^2 + m_{t^c}^2 + m_Q^2 + A_t^2) - 8C_L^{MM}.
\end{aligned} \tag{3.17}$$

Equation (3.9) now becomes

$$\begin{aligned}
H_{t^c} &= 4A_t\lambda_t^2 + 4C_{t^c}^M,\\
H_{b^c} &= 4A_b\lambda_b^2 + 4C_{b^c}^M,\\
H_Q &= 2(A_t\lambda_t^2 + A_b\lambda_b^2) + 4C_Q^M,\\
H_{\tau^c} &= 4A_\tau\lambda_\tau^2 + 4C_{\tau^c}^M,\\
H_L &= 2A_\tau\lambda_\tau^2 + 4C_L^M,\\
H_{u^c} &= 4C_{t^c}^M,\\
H_{d^c} &= 4C_{b^c}^M,\\
H_R &= 4C_Q^M,\\
H_{e^c} &= 4C_{\tau^c}^M,\\
H_N &= 4C_L^M,\\
H_{H_1} &= 6A_b\lambda_b^2 + 2A_\tau\lambda_\tau^2 + 4C_L^M,\\
H_{H_2} &= 6A_t\lambda_t^2 + 4C_L^M.
\end{aligned} \tag{3.18}$$

Equations (3.11), (3.12) now become

$$16\pi^2\hat{\beta}_\xi^{(1)}=2\sqrt{\tfrac{3}{5}}g_1(m_Q^2+2m_R^2-m_L^2-2m_N^2-2m_{t^c}^2-4m_{u^c}^2 +m_{b^c}^2+2m_{d^c}^2+m_{\tau^c}^2+2m_{e^c}^2-m_{H_1}^2+m_{H_2}^2), \tag{3.19}$$

$$(16\pi^2)^2\hat{\beta}_\xi^{(2)}=-4\sqrt{\tfrac{3}{5}}g_1(m_Q^2P_Q+2m_R^2P_R-m_L^2P_L-2m_N^2P_N -2m_{t^c}^2P_{t^c}-4m_{u^c}^2P_{u^c}+m_{b^c}^2P_{b^c}+2m_{d^c}^2P_{d^c} +m_{\tau^c}^2P_{\tau^c}+2m_{e^c}^2P_{e^c}-m_{H_1}^2P_{H_1}+m_{H_2}^2P_{H_2}). \tag{3.20}$$

Finally, Eq. (3.14) is replaced by

$$\beta_{\xi_1}^{(3)}=m_Q^2\gamma_Q^{(2)}+2m_R^2\gamma_R^{(2)}-m_L^2\gamma_L^{(2)}-2m_N^2\gamma_N^{(2)}-2m_{t^c}^2\gamma_{t^c}^{(2)} -4m_{u^c}^2\gamma_{u^c}^{(2)}+m_{b^c}^2\gamma_{b^c}^{(2)}+2m_{d^c}^2\gamma_{d^c}^{(2)}+m_{\tau^c}^2\gamma_{\tau^c}^{(2)} +2m_{e^c}^2\gamma_{e^c}^{(2)}-m_{H_1}^2\gamma_{H_1}^{(2)}+m_{H_2}^2\gamma_{H_2}^{(2)}$$

$$\beta_{\xi_2}^{(3)}=W_QP_Q+2W_RP_R-W_LP_L-2W_NP_N-2W_{t^c}P_{t^c} -4W_{u^c}P_{u^c}+W_{b^c}P_{b^c}+2W_{d^c}P_{d^c}+W_{\tau^c}P_{\tau^c} +2W_{e^c}P_{e^c}-W_{H_1}P_{H_1}+W_{H_2}P_{H_2}$$

$$\beta_{\xi_3}^{(3)}=H_Q^2+2H_R^2-H_L^2-2H_N^2-2H_{t^c}^2-4H_{u^c}^2+H_{b^c}^2+2H_{d^c}^2 +H_{\tau^c}^2+2H_{e^c}^2-H_{H_1}^2+H_{H_2}^2$$

$$\beta_{\xi_4}^{(3)}=m_Q^2P_Q^2+2m_R^2P_R^2-m_L^2P_L^2-2m_N^2P_N^2-2m_{t^c}^2P_{t^c}^2 -4m_{u^c}^2P_{u^c}^2+m_{b^c}^2P_{b^c}^2+2m_{d^c}^2P_{d^c}^2+m_{\tau^c}^2P_{\tau^c}^2 +2m_{e^c}^2P_{e^c}^2-m_{H_1}^2P_{H_1}^2+m_{H_2}^2P_{H_2}^2$$

$$\beta_{\xi_5}^{(3)}=W_QC_Q+2W_RC_Q-W_LC_L-2W_NC_L-2W_{t^c}C_{t^c} -4W_{u^c}C_{t^c}+W_{b^c}C_{b^c}+2W_{d^c}C_{b^c}+W_{\tau^c}C_{\tau^c} +2W_{e^c}C_{\tau^c}-W_{H_1}C_L+W_{H_2}C_L$$

$$\beta_{\xi_6}^{(3)}=2(H_QC_Q^M+2H_RC_Q^M-H_LC_L^M-2H_NC_L^M-2H_{t^c}C_{t^c}^M -4H_{u^c}C_{t^c}^M+H_{b^c}C_{b^c}^M+2H_{d^c}C_{b^c}^M+H_{\tau^c}C_{\tau^c}^M+2H_{e^c}C_{\tau^c}^M -H_{H_1}C_L^M+H_{H_2}C_L^M)$$

$$\beta_{\xi_7}^{(3)}=3(C_Q^{MM}C_Q-C_L^{MM}C_L-2C_{t^c}^{MM}C_{t^c}+C_{b^c}^{MM}C_{b^c} +C_{\tau^c}^{MM}C_{\tau^c})$$

$$\beta_{\xi_8}^{(3)}=3[|C_Q^M|^2-|C_L^M|^2-2|C_{t^c}^M|^2+|C_{b^c}^M|^2+|C_{\tau^c}^M|^2]. \tag{3.21}$$

IV. RUNNING ANALYSIS

As we mentioned in the Introduction, if we have no prejudice as to the value of ξ at the gauge unification scale M_X, then we may as well treat ξ as a free parameter at the weak scale [3], and the running of ξ is irrelevant. However, it is conceivable that the underlying theory at scales beyond M_X may favor certain values of $\xi(M_X)$, and then the running of ξ would need to be considered. We shall see that for currently popular choices of boundary conditions at M_X—namely, the minimal supergravity scenario and the anomaly mediated supersymmetry breakdown (AMSB) scenario—the running of ξ is determined predominantly by the first term on the right-hand side of Eq. (1.3) between M_X and M_Z, and hence to a good approximation we have

$$\xi(M_Z)\approx\frac{g_1(M_Z)}{g_1(M_X)}\xi(M_X). \tag{4.1}$$

For instance, we find from Eqs. (2.7), (2.8) that universal soft masses at M_X imply $\hat{\beta}_\xi^{(1)}(M_X)=\hat{\beta}_\xi^{(2)}(M_X)=0$, using Eq. (2.6), and the fact that it follows immediately from Eq. (2.2b) using gauge invariance and anomaly cancellation [Eq. (2.5)] that

$$\mathrm{Tr}[\mathcal{Y}\gamma^{(1)}]=0. \tag{4.2}$$

Moreover, it is easy to show, using the result for $\beta_{m^2}^{(1)}$ from Ref. [6], that if we work consistently at one loop, then $\mathrm{Tr}[\mathcal{Y}m^2]$ is scale invariant. So if initially $\xi=\mathrm{Tr}[\mathcal{Y}m^2]=0$, then ξ remains zero under (one loop) renormalization group (RG) evolution. With typical universal conditions at M_X with soft masses m_0 and $M\sim m_0$, $A\sim m_0$, we find (using three loops for β_ξ and two loops for the other β functions) that $\xi\approx 0.001m_0^2$ at M_Z.

Another favored set of boundary conditions is those derived from AMSB [9]. Here the soft masses are given by

$$(m^2)^i{}_j=\frac{1}{2}|m_{3/2}|^2\mu\frac{d\gamma^i{}_j}{d\mu}, \tag{4.3}$$

where $m_{3/2}$ is the gravitino mass. In fact, since the AMSB result is RG invariant, it applies at all scales between M_X and M_Z. We then find from Eqs. (2.7), (2.8) that up to two loops, we may write

$$16\pi^2\hat{\beta}_\xi=g_1|m_{3/2}|^2\mu\frac{d}{d\mu}\mathrm{Tr}[\mathcal{Y}(\gamma-\gamma^2)]. \tag{4.4}$$

Gauge invariance and anomaly cancellation combined with Eqs. (2.2b) and (2.3b) yield [1]

$$\mathrm{Tr}[\mathcal{Y}\gamma^{(1)}]=\mathrm{Tr}[\mathcal{Y}(\gamma^{(2)}-(\gamma^{(1)})^2)]=0, \tag{4.5}$$

and so $\hat{\beta}_\xi$ vanishes through two loops. Therefore to a good approximation $\xi(M_Z)$ will be given by Eq. (4.1), and once again will be negligible at M_Z if it is zero at M_X.

However, if non-universal scalar masses at M_X are contemplated, then the effects of $\hat{\beta}_\xi$ might be significant—as

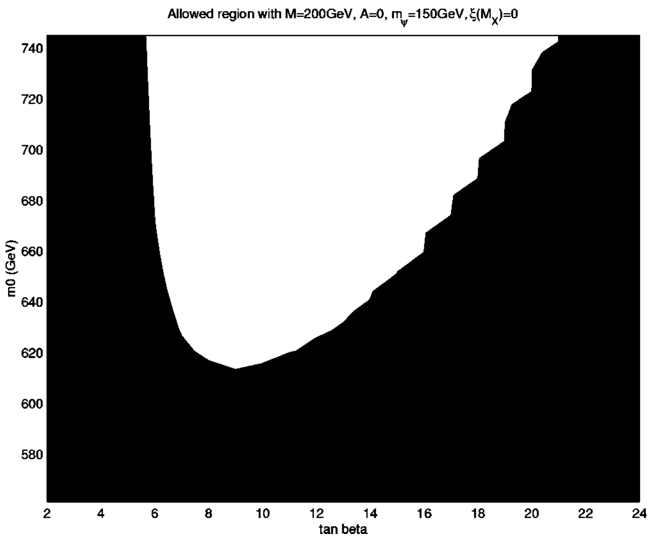

FIG. 1. The region of the $m_0 \tan\beta$ plane corresponding to an acceptable electroweak vacuum, for $M=200$ GeV, $m_\psi=150$ GeV, $A=0$ and $\xi(M_X)=0$. The shaded region corresponds to one or more sparticle or Higgs boson masses in violation of current experimental bounds.

was noted in Ref. [10], for instance. Another context where $\hat{\beta}_\xi$ might play a role is that of non-standard soft supersymmetry breaking [11]. This is because with the non-standard terms (for example $\phi^2\phi^*$ terms) the result that $\mathrm{Tr}[\mathcal{Y}m^2]$ is one-loop scale invariant is not preserved. It follows that even with universal boundary conditions for m^2 and $\xi=0$ at M_X, ξ becomes non-zero at M_Z even with one-loop running. In the current context of the MSSM with the 3rd generation dominating, the additional soft terms are given by

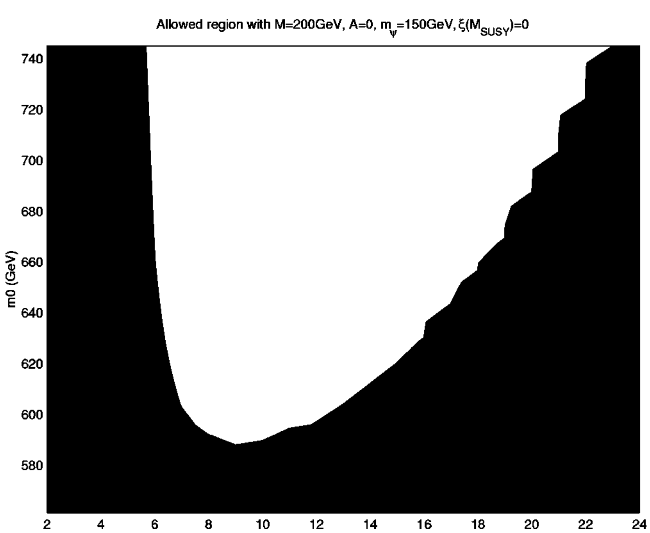

FIG. 2. The region of the $m_0 \tan\beta$ plane corresponding to an acceptable electroweak vacuum, for $M=200$ GeV, $m_\psi=150$ GeV, $A=0$ and $\xi(M_{\mathrm{SUSY}})=0$. The shaded region corresponds to one or more sparticle or Higgs boson masses in violation of current experimental bounds.

TABLE I. Spectra (in GeV) for $\xi(M_X)=0$ and for $\xi(M_{\mathrm{SUSY}})=0$, with $M=200$ GeV, $m_0=640$ GeV, $A=0$, $m_\psi=150$ GeV at M_X, and with $\tan\beta=8$.

	$\xi(M_X)=0$	$\xi(M_{\mathrm{SUSY}})=0$
$\tilde{t}_1$	639	649
$\tilde{t}_2$	319	334
$\tilde{b}_1$	604	615
$\tilde{b}_2$	776	780
$\tilde{\tau}_1$	625	639
$\tilde{\tau}_2$	663	658
$\tilde{u}_L$	792	794
$\tilde{u}_R$	793	785
$\tilde{d}_L$	796	798
$\tilde{d}_R$	781	785
$\tilde{e}_L$	664	657
$\tilde{e}_R$	632	646
$\tilde{\nu}_\tau$	657	650
$\tilde{\nu}_e$	659	652
h	116	116
H	231	291
A	230	290
$H^\pm$	244	302
$\tilde{\chi}_1^\pm$	120	120
$\tilde{\chi}_2^\pm$	201	201
$\tilde{\chi}_1$	68	68
$\tilde{\chi}_2$	116	116
$\tilde{\chi}_3$	167	167
$\tilde{\chi}_4$	234	234
$\tilde{g}$	521	521

$$L_{\mathrm{soft}}^{\mathrm{new}}=m_\psi\psi_{H_1}\psi_{H_2}+\overline{A}_t\lambda_t H_1^* Q t^c+\overline{A}_b\lambda_b H_2^* Q b^c + \overline{A}_\tau\lambda_\tau H_2^* L\tau^c+\mathrm{H.c.} \tag{4.6}$$

Now in Ref. [11] we assumed, in fact, that ξ was zero at M_Z; here we explore the more natural assumption that $\xi=0$ at the unification scale. We follow Ref. [11] in dropping the explicit μ term from the superpotential, since it can be subsumed into $L_{\mathrm{soft}}^{\mathrm{new}}$. With given values at M_X for m_ψ and for the universal parameters A, M and m_0, and for a given $\tan\beta$, we adjust $\overline{A}_t=\overline{A}_b=\overline{A}_\tau=\overline{A}$ (at M_X) to obtain an acceptable electroweak vacuum. As in Ref. [11], we have made allowance for radiative corrections by using the tree Higgs minimization conditions, but evaluated at the scale $M_{\mathrm{SUSY}}\approx m_0$. In Fig. 1 we show (for illustrative values of M, m_ψ and A) the region of the $m_0 \tan\beta$ plane where this can be achieved.

For comparison, we show in Fig. 2 the corresponding region for $\xi(M_{\mathrm{SUSY}})=0$. We notice that it is qualitatively similar, though slightly larger.

Note that this figure differs slightly from Fig. 1 of Ref. [11]. This is because we have incorporated one-loop corrections to the Higgs boson mass and because we have taken

TABLE II. Values for $\xi(M_{\rm SUSY})$ with $\xi(M_X)=0$ and with $M=200$ GeV, $A=0$ and $m_\psi=150$ GeV at M_X.

m_0(GeV)	$\tan\beta$	$\xi(M_{\rm SUSY})$ (GeV)^2
640	8	-5.07×10^4
700	6	-5.48×10^4
700	8	-5.02×10^4
700	16	-5.15×10^4
800	6	-5.61×10^4
800	8	-4.90×10^4
800	16	-4.75×10^4

account of the increasingly stringent experimental bounds (in particular increasing m_ψ at M_X to get acceptable chargino masses). For $m_0=640$ GeV and $\tan\beta=8$, we find $\overline{A}=1.07(1.01)$ TeV, $\overline{A}_t(M_{\rm SUSY})\approx 661(627)$ GeV, $\overline{A}_\tau(M_{\rm SUSY})\approx 664(630)$ GeV, $\overline{A}_b(M_{\rm SUSY})\approx 491(469)$ GeV. [The pairs of numbers correspond to $\xi(M_X)=0$, $\xi(M_{\rm SUSY})=0$ respectively.] The spectra obtained for $\xi(M_X)=0$ and for $\xi(M_{\rm SUSY})=0$ are given in Table I. We see that there are significant differences, especially in the masses of *H*, *A* and $H^\pm$. On the other hand, the chargino and neutralino masses are unaffected, with a lightest supersymmetric particle (LSP) neutralino.

Finally, in Table II we give the values of $\xi(M_{\rm SUSY})$ for some typical points in the allowed region of Fig. 1. We see indeed that $\xi(M_{\rm SUSY})$ is quite sizable.

We have verified that the same results are obtained if we either (1) perform the RG evolution in the ξ-uneliminated theory and then eliminate ξ (via its equation of motion) at low energies or (2) eliminate ξ at M_X, and evolve to low energies with the (modified) ξ-eliminated β functions. For a general discussion of the equivalence of these procedures, see Refs. [1,2].

V. CONCLUSIONS

In this paper we have extended the results of Ref. [2] for the renormalization of the Fayet-Iliopoulos *D* term to the case of a direct product gauge group, and applied the result to the MSSM. With standard soft supersymmetry breaking and universal boundary conditions at M_X, then ξ is negligible at low energies if $\xi(M_X)=0$. However, with non-standard soft breakings (and/or non-universal boundary conditions for the standard ones) we find significant effects even for $\xi(M_X)=0$. In the non-standard breaking case, the effect is especially marked for the masses of *H*, *A* and $H^\pm$, which decrease significantly when ξ is taken into account.

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APPENDIX: GENERAL RESULT FOR SEVERAL ABELIAN FACTORS

In this appendix we give the general results for the case of a direct product group with several Abelian factors. As we mentioned earlier, the situation is complicated by the possibility of ‘‘kinetic mixing’’ [4] between the different Abelian factors. We can accommodate this possibility by introducing a matrix of couplings for the Abelian factors. Suppose that the gauge group is $(U_1)^a\Pi^n_{\alpha=a+1}G_\alpha$, where the G_α, $\alpha=a+1,\dots,n$ are non-Abelian. The gauge couplings are then $g_{\alpha\beta}$, where $g_{\alpha\beta}=g_\alpha\delta_{\alpha\beta}$, $\alpha=a+1\dots n$, with a similar form for the gauge β functions. The gaugino masses also form a matrix $M_{\alpha\beta}$ with an analogous structure, as do their β functions. Suppose the hypercharges of the Abelian factors for a given representation are $\mathcal{Y}_\alpha$, $\alpha=1,\dots,a$. Then we define

$$\overline{\mathcal{Y}}_\alpha=\sum_{\beta=1}^{a}\mathcal{Y}_\beta g_{\beta\alpha},\quad \alpha=1,\dots,a, \tag{A1}$$

and a generalized quadratic Casimir matrix

$$\overline{C}(R)=\sum_{\alpha=1}^{a}\overline{\mathcal{Y}}_\alpha\overline{\mathcal{Y}}_\alpha+\sum_{\alpha=a+1}^{n}g_\alpha^2C(R_\alpha). \tag{A2}$$

The Fayet-Iliopoulos couplings now form a vector ξ_α, $\alpha=1,\dots,a$, and we have the matrix equation

$$\beta_\xi=g^{-1}\beta^g\xi+\hat{\beta}_\xi. \tag{A3}$$

We can now give the explicit general results, starting with the gauge β functions and anomalous dimension. At one loop,

$$16\pi^2\beta^{g(1)}=g\overline{Q} \tag{A4}$$

where

$$\overline{Q}_{\alpha\beta}=\mathrm{Tr}[\overline{\mathcal{Y}}_\alpha\overline{\mathcal{Y}}_\beta],\quad \alpha,\beta=1,\dots,a,$$
$$\overline{Q}_{\alpha\beta}=g_\alpha^2Q_\alpha\delta_{\alpha\beta},\quad \alpha=a+1,\dots,n, \tag{A5}$$

and

$$16\pi^2(\gamma^{(1)})^i{}_j=P^i{}_j\equiv\tfrac{1}{2}Y^{ikl}Y_{jkl}-2\overline{C}(R)^i{}_j. \tag{A6}$$

At two loops,

$$(16\pi^2)^2(\gamma^{(2)})^i{}_j=-[Y_{jmn}Y^{mpi}+2\overline{C}(R)^p{}_j\delta^i{}_n]P^n{}_p+2(\overline{Q}_{\alpha\beta}\overline{\mathcal{Y}}_\alpha\overline{\mathcal{Y}}_\beta+g_\alpha^4Q_\alpha C(R_\alpha))^i{}_j \tag{A7}$$

and

$$(16\pi^2)^2(\beta^{g(2)})_{\alpha\beta}=-2g_{\alpha\gamma}\mathrm{Tr}[P\overline{\mathcal{Y}}_\gamma\overline{\mathcal{Y}}_\beta],\quad \alpha,\beta=1,\dots,a,$$
$$(16\pi^2)^2(\beta^{g(2)})_\alpha=2g_\alpha^5C(G_\alpha)Q_\alpha-2g_\alpha^3r_\alpha^{-1}\mathrm{Tr}[PC(R_\alpha)],$$
$$\alpha=a+1,\dots,n. \tag{A8}$$

At three loops we have

$$
\begin{aligned}
(16\pi^2)^3(\beta^{g\mathrm{DRED}(3)})_{\alpha\beta}=g_{\alpha\gamma}\Bigg\{&3Y^{ikm}Y_{jkn}P^n{}_m(\overline{\mathcal{Y}}_\gamma\overline{\mathcal{Y}}_\beta)^j{}_i+6\mathrm{Tr}[P\overline{\mathcal{Y}}_\gamma\overline{\mathcal{Y}}_\beta\overline{C}(R)]+\mathrm{Tr}[P^2\overline{\mathcal{Y}}_\gamma\overline{\mathcal{Y}}_\beta]\\
&-6\sum_{\kappa,\lambda=1}^{a}\overline{Q}_{\kappa\lambda}\mathrm{Tr}[\overline{\mathcal{Y}}_\gamma\overline{\mathcal{Y}}_\beta\overline{\mathcal{Y}}_\kappa\overline{\mathcal{Y}}_\lambda]-6\sum_{\kappa=a+1}^{n}g_\kappa^4Q_\kappa\mathrm{Tr}[\overline{\mathcal{Y}}_\gamma\overline{\mathcal{Y}}_\beta C(R_\kappa)]\Bigg\},\quad \alpha,\beta=1,\dots,a,\\
(\beta^{g\mathrm{DRED}(3)})_\alpha=\;&3r_\alpha^{-1}g_\alpha^3Y^{ikm}Y_{jkn}P^n{}_mC(R_\alpha)^j{}_i+6r_\alpha^{-1}g_\alpha^3\mathrm{Tr}[PC(R_\alpha)\overline{C}(R)]+r_\alpha^{-1}g_\alpha^3\mathrm{Tr}[P^2C(R_\alpha)]\\
&-6r_\alpha^{-1}g_\alpha^3\sum_{\kappa,\lambda=1}^{a}\overline{Q}_{\kappa\lambda}\mathrm{Tr}[C(R_\alpha)\overline{\mathcal{Y}}_\kappa\overline{\mathcal{Y}}_\lambda]-6r_\alpha^{-1}g_\alpha^3\sum_{\kappa=a+1}^{n}g_\kappa^4Q_\kappa\mathrm{Tr}[C(R_\alpha)C(R_\kappa)]\\
&-4r_\alpha^{-1}g_\alpha^5C(G_\alpha)\mathrm{Tr}[PC(R_\alpha)]+g_\alpha^7Q_\alpha C(G_\alpha)[4C(G_\alpha)-Q_\alpha],\quad \alpha=a+1,\dots,n.
\end{aligned}\tag{A9}
$$

For the Fayet-Iliopoulos couplings we have, at one loop,

$$16\pi^2[\hat{\beta}_\xi^{(1)}]_\alpha=\mathrm{Tr}[\overline{\mathcal{Y}}_\alpha m^2],\quad \alpha=1,\dots,a,\tag{A10}$$

and, at two loops,

$$16\pi^2[\hat{\beta}_\xi^{(2)}]_\alpha=-4\mathrm{Tr}[\overline{\mathcal{Y}}_\alpha m^2\gamma^{(1)}].\tag{A11}$$

Finally,

$$
\begin{aligned}
(16\pi^2)^3(\hat{\beta}_\xi^{(3)\mathrm{DRED}'})_\alpha=\;&-6(16\pi^2)^2\mathrm{Tr}[\overline{\mathcal{Y}}_\alpha m^2\gamma^{(2)}]-4\mathrm{Tr}[\overline{\mathcal{Y}}_\alpha WP]-\frac{5}{2}\mathrm{Tr}[\overline{\mathcal{Y}}_\alpha HH^\dagger]+2\mathrm{Tr}[\overline{\mathcal{Y}}_\alpha P^2m^2]\\
&-24\zeta(3)\mathrm{Tr}[\overline{\mathcal{Y}}_\alpha W\overline{C}(R)]+12\zeta(3)\mathrm{Tr}[\overline{\mathcal{Y}}_\alpha H\overline{C}^{M}{}^*(R)+\mathrm{c.c.}]-96\zeta(3)\mathrm{Tr}[\overline{\mathcal{Y}}_\alpha\overline{C}^{MM^*}(R)\overline{C}(R)]\\
&-24\zeta(3)\{\mathrm{Tr}[\overline{\mathcal{Y}}_\alpha\overline{C}^M(R)\overline{C}^{M}{}^*(R)]+\mathrm{c.c.}\},
\end{aligned}\tag{A12}
$$

where

$$
\begin{aligned}
W^i{}_j&=\left(\frac{1}{2}Y^2m^2+\frac{1}{2}m^2Y^2+h^2\right)^i{}_j+2Y^{ipq}Y_{jpr}(m^2)^r{}_q-8\overline{C}^{MM^*}(R)^i{}_j,\\
H^i{}_j&=h^{ikl}Y_{jkl}+4\overline{C}^M(R)^i{}_j,\\
\overline{C}^M(R)&=\sum_{\alpha,\beta=1}^{a}M_{\alpha\beta}\overline{\mathcal{Y}}_\alpha\overline{\mathcal{Y}}_\beta+\sum_{\alpha=a+1}^{n}g_\alpha^2M_\alpha C(R_\alpha),\\
\overline{C}^{MM^*}(R)&=\sum_{\alpha,\beta=1}^{a}(MM^*)_{\alpha\beta}\overline{\mathcal{Y}}_\alpha\overline{\mathcal{Y}}_\beta+\sum_{\alpha=a+1}^{n}M_\alpha M_\alpha^*g_\alpha^2C(R_\alpha).
\end{aligned}\tag{A13}
$$

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